ON RESULTS OF COMPARISON OF A FINITE-DIFFERENCE METHOD AND THE METHOD OF STEEPEST DESCENTS FOR THE ANALYSIS OF TRANSIENT STRESS WAVES IN PLATES

(O RESUL'TATAKH SOPOSTAVLENIIA METODA SETOK I METODA PEREVALA PRI ANALIZE PEREKHODNOGO VOLNOVOGO PROTSESSA DEFORMATSII PLIT)

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A number of results is presented in [1] indicating the regions of validity and effective application of approximate theories and methods for the analysis of transient dynamic deformations of plates. In this paper additional information is given on the boundary between the regions of effective applicability of two methods of integrating the equations of a Timoshenko type theory for the title problem. Conclusions are presented concerning the accuracy of these methods for the various quantities which are computed.

1. The problem is one of transient stress waves in a plate. The deformations, which will depend on one coordinate of the middle surface of the plate, are caused by the action of a load applied along some straight line or in a small region of the surface of the plate. Under the assumption that the object of the study consists in determining the aspects of the wave motion which affect the magnitude of the amplitudes most strongly, rough boundaries were determined in [1] for the regions of validity and effective applicability of approximate methods of integration of the equations of the linear theory of elasticity.

In particular, it was established that after the initial front has traveled a distance equal to several plate thicknesses, it is permissible to replace the equations of the theory of elasticity by the equations of a theory of Timoshenko type in the region behind the nominal "front" of the Rayleigh surface waves, and in certain cases also in a small region ahead of that front (the immediate vicinity of the position of the applied load may be an exception). The region of validity for application of a theory of Timoshenko type includes the region of applicability of the Kirchhoff theory, but is considerably broader than the latter [1 and 2]. This region is characterized by large displacements and frequently is of greatest interest in the solution of particular problems. The situation is similar in the case of axisymmetric problems of analysis of cylindrical and spherical shells. Therefore, it is important to have available an effective system of methods of integration of the equations of a theory of Timoshenko type. To develop such a theory, information is needed on the accuracy and effectiveness of the computational schemes which have been proposed (see [3]).

We shall examine a system consisting of the following approximate methods of integration of the equation of a theory of Timoshenko type: A – the computation of the contour integrals for the inversion of the Laplace transforms by the method of steepest descents, which provides an asymptotic approximation for the solution for time $t \to \infty$ at points sufficiently far from the fronts; B — an asymptotic expansion near the fronts based on the expansion of the Laplace transforms in negative integral powers of s (where s is the variable of the Laplace transform); C — an improved variant of the finite-difference method in which, with aid of B, partial solutions which include the discontinuities of the unknown quantities are separated out, and the finite-difference method is applied to compute only that part of the solution which is continuous along with the first and second derivatives occurring in the computations.

Methods A and B were first introduced in [4] for the analysis of a beam; method C was proposed and used in [5 and 6] for the analysis of plates and a spherical shell.

Agreement of the results in methods B and C is guaranteed automatically in the regions near the fronts. However, comparison of the results of methods A and C is of interest for two reasons. Firstly, the effectiveness of the system of methods of integration A, B, C depends greatly on the value of time $t = t_1$ at which we switch from method A to method C. For in the analysis of an infinite or semi-infinite body (i.e. before reflection of waves from the supports) the amount of calculation in method C increases proportionally to t_1^2, while for a finite body method A has practical value only provided that t_1 is much smaller than the time of transit of the elastic waves across a characteristic dimension of the middle surface of the plate. Secondly, the application of the asymptotic solution of method A for $t \to \infty$ deserves attention as a means of verifying the reliability of the finite-difference method C, which has been only recently introduced [5 and 6].

The comparisons, [1], of displacements computed by methods A and C for a semi-infinite plate, loaded by a uniformly distributed suddenly applied edge moment, showed that the results agree fairly well only after the elastic waves have traversed a distance equal to several dozen plate thicknesses, and from a practical point of view did not provide any information on the reliability of method B. The plan dimensions of plates are often of the order of a few dozen plate thicknesses (or less). Therefore, the results of [1] which have been cited cast doubt on the practical value of method A, not only for the integration of equations of a theory of Timoshenko type, but also for integration of the equations of elasticity, as carried out, for example, in [2].

It is shown below that comparison of curves of bending moment M and shear Q computed by methods A and C give much more favorable results for method A and allow one to conclude that method C is reliable.

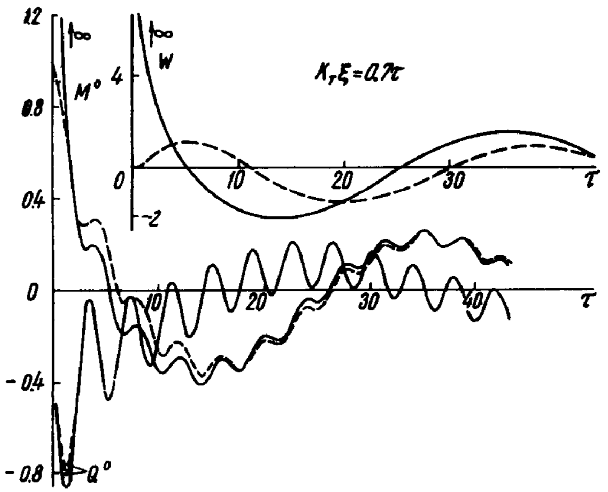

Fig. 1

We shall present a mathematical statement of the problem and also explain the notation in Fig.1. Detailed descriptions of methods A and C are omitted, inasmuch as they are given in [2 and 5], respectively.

2. Let E be the modulus of elasticity, ν be Poisson's ratio, ρ be the density of the material, $2h$ the thickness of the plate; c_{Π} and c_{T} are the velocities of propagation of the first and second fronts in a theory of Timoshenko type, k_T^2 is the shear coefficient, τ the dimensionless time, ξ the dimensionless longitudinal coordinate (divided by h), W is the dimensionless average normal displacement (divided by h), ψ is the angle of rotation, M and Q are the bending moment and shear, $\partial_\xi, \partial_\tau$ are differentiation operators with respect to ξ and τ, $H(\tau)$ is the Heaviside unit step function, and l_ξ and l_τ are the dimensionless finite-difference grid spacings. We have

$$c_{\Pi}=\left(\frac{E}{\rho(1-\nu^2)}\right)^{1/2}, \quad c_{T}=\left(\frac{Ek_T^2}{2\rho(1+\nu)}\right)^{1/2}, \quad k=\frac{c_T}{c_{\Pi}}, \quad \tau=\frac{c_T t}{h} \tag{2.1}$$

$$M=\frac{Eh^2}{1+\nu}M^\circ, \qquad Q=\frac{Eh}{1+\nu}Q^\circ \tag{2.2}$$

$$M^\circ=-\frac{2}{3(1-\nu)}\partial_\xi\psi, \qquad Q^\circ=k_T^2(\partial_\xi W-\psi) \tag{2.3}$$

The equations of motion of the Timoshenko type theory are expressed in the form

$$(\partial_\xi^2-\partial_\tau^2)W-\partial_\xi\psi=0, \qquad 3\partial_\xi W+(k^{-2}\partial_\xi^2-\partial_\tau^2-3)\psi=0 \tag{2.4}$$

We shall examine the outward going stress waves in an infinite plate caused by a suddenly applied edge moment which is uniformly distributed along the edge $\xi = 0$. We apply zero initial conditions; the boundary conditions are

$$M^\circ(0,\tau)=\frac{2C}{3-3\nu}H(\tau), \qquad W(0,\tau)=0, \qquad C=\text{const} \tag{2.5}$$

and the numerical coefficients are $\nu=0.3$, $k_T^2=0.860$, $C=1$.

3. If the Laplace transform is defined by Equations

$$\int_0^\infty F(\xi,\tau)e^{-s\tau}d\tau=F^L(\xi,s), \qquad F(\xi,\tau)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}F^L(\xi,s)e^{s\tau}ds \tag{3.1}$$

then the transform of the solution of the problem under study is as follows:

$$W^L=C\sum_{j=1}^{2}B_{Wj}e^{-\lambda_j\xi}, \qquad \psi^L=C\sum_{j=1}^{2}B_{\psi j}e^{-\lambda_j\xi} \tag{3.2}$$

where

$$B_{Wj}=-\frac{(-1)^j}{s(\lambda_1^2-\lambda_2^2)}, \qquad B_{\psi j}=\frac{(-1)^j(s^2-\lambda_j^2)}{s\lambda_j(\lambda_1^2-\lambda_2^2)} \tag{3.3}$$

$$\lambda_j=\left(\frac{s^2}{2}\left[(1+k^2)\pm(1-k^2)\left(1-\frac{12k^2}{s^2(1-k^2)^2}\right)^{1/2}\right]\right)^{1/2} \qquad (j=1,2) \tag{3.4}$$

In order to compute the λ_j from Equation (3.4), the signs must be chosen so that $\operatorname{Re}\lambda_j > 0$ for $\operatorname{Re} s > 0$. Expressions for $M^{\circ L}$ and $Q^{\circ L}$ are easily obtained from (2.3), (3.2) and (3.3).

The expressions (3.2) for the transforms cannot be inverted exactly. However, an asymptotic approximation for $\tau \to \infty$ can easily be constructed by the method of steepest descents (method A) by using the saddle points on the imaginary axis $s = i\omega$ (where ω is a real quantity). For each particular ray $\xi/\tau = r = \text{const} < 1$ in the τ, ξ plane there exist two pairs of saddle points located symmetrically with respect to the point $s = 0$. The calculations were performed using the formulas for the first approximation in the method of steepest descents and the additional numerical information given in [2]. The results for W, N° and Q° on the ray $k_T r = 0.7$ are shown in Fig.1 by solid lines. We remark that consideration was also given to other rays which belong to that part of the region of applicability of a theory of Timoshenko type where application of the Kirchhoff theory is not justified.

4. In order to carry out method C, studies were first made to find the frontal discontinuities. The following partial solutions which contain the the frontal discontinuities were found by applying method B (i.e. by expanding the expressions for $B_{Wj}, B_{\psi j}, \lambda_j$ in negative integral powers of s) :

$$W_0 = \left[\frac{(\tau - k\xi)^2}{2(1-k^2)} H(\tau - k\xi) - \frac{(\tau-\xi)^2}{2(1-k^2)} H(\tau-\xi)\right] C, \qquad \psi_0 = \frac{\tau - k\xi}{k} H(\tau - k\xi)\, C \tag{4.1}$$

The complete solution was constructed in the form

$$W = W_0 + W_1, \qquad \psi = \psi_0 + \psi_1 \tag{4.2}$$

in which the functions W_1, ψ_1 which are continuous together with their first and second derivatives, were calculated by the finite-difference method in the τ, ξ plane. At the internal points of the region $0 < \xi < \tau k^{-1}$, with the exception of the first points behind the front $\xi = \tau k^{-1}$, the quantities W_1, ψ_1 were calculated from the inhomogeneous equations obtained from (2.4) by the substitution of Equation (4.2). At the edge $\xi = 0$, W_1 and ψ_1 were determined on the basis of the conditions (2.5), and on the front $\xi = \tau k^{-1}$ and at the first points behind this front by the frontal conditions

$$W_1 = \psi_1 = 0, \quad \partial_\xi W_1 = \partial_\xi \psi_1 = 0 \quad \text{for} \quad \xi = \tau k^{-1} \tag{4.3}$$

The dimensionless grid spacings were taken as $l_\xi = 0, 1$, $l_\tau = {}^1/_2 k l_\xi$ in the calculations.

With the aid of Equation (2.3) N° and Q° were found from W and ψ. The results of the analysis are shown in Fig.1 by dashed lines.

5. The results of the calculations show (see Fig.1) that the N° and Q° curves computed according to methods A and C agree after the elastic waves travel a distance equal to several plate thicknesses, the agreement improving as τ increases. Close agreement can be expected for W only for much larger values of τ.

Analysis of these results permit us to draw the following general conclusions.

1) The improved variant of the finite-difference method that was proposed in [5 and 6] (Method C) is reliable.

2) Method A (use of the formulas of the first approximation in the method of steepest descents to evaluate the contour integrals of the formal solution) has quite different regions of applicability for different computed quantities.

3) The great effectiveness of method A for the calculation of N° and Q° compared to its effectiveness for finding W is not peculiar to just the problem studied. The reson for this phenomenon is as follows. The path of integration used in calculating the contour integrals for the inversion can be considered to consist of two parts: (a) the part passing through the saddle points, (b) the part encircling the singularities. Method A approximates the contribution of part (a), but does not take into account the contribution of part (b). The latter contribution is less important for the more rapidly oscillating quantities (N° and Q°).

In connection with conclusions (2) and (3), it is curious to note that, within the limits of the Kirchhoff theory, the Laplace transform inversion integrals may be evaluated exactly for the present problem. For this case, method A results in the exact expression for Q°; it gives a good approximation for N° even for small τ, but yields a practically applicable approximation for normal displacement only for very large τ.

The first author is responsible for programing the calculations on the "Minsk-2" electronic digital computer; the second author is responsible for the theoretical part of the study.

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